4. Berwick's Problem of the Seven Sevens

In the following division problem, in which the divisior goes into the dividend without a remainder, the numbers that occupied the places marked with the asterisks were accidentally erased. What are the missing numbers?

$$
\begin{array}{cccccccccc l}
\ast & \ast & 7 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \div\ \ast\ \ast\ \ast\ \ast\ 7\ \ast\ =\ \ast\ \ast\ 7\ \ast\ \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & & & & & \\ \cline{1-6}
\ast & \ast & \ast & \ast & \ast & 7 & \ast & & & & \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & & & & \\ \cline{1-7}
 & & \ast & 7 & \ast & \ast & \ast & \ast & & & \\
 & & \ast & 7 & \ast & \ast & \ast & \ast & & & \\ \cline{3-8}
 & & \ast & \ast & \ast & \ast & \ast & \ast & \ast & & \\
 & & \ast & \ast & \ast & \ast & 7 & \ast & \ast & & \\ \cline{3-9}
 & & & & \ast & \ast & \ast & \ast & \ast & \ast & \\
 & & & & \ast & \ast & \ast & \ast & \ast & \ast & \\ \cline{5-10}
\end{array}
$$

This remarkable problem comes from the English mathematician E.H.Berwick, who published it in 1906 in the periodical *The School World.*

Solution. We will assign a separate letter to each of the missing numerals. In addition we set ∂ equal to the divisor. The problem then has the following appearance:

$$
\begin{array}{cccccccccc l}
A & B & 7 & C & D & E & L & Q & W & z & \div\ \alpha\ \beta\ \gamma\ \delta\ 7\ \varepsilon\ =\ \kappa\ \lambda\ 7\ \mu\ \nu \\
a & b & \Delta & c & d & e & & & & & \\ \cline{1-6}
F & G & H & I & K & 7 & L & & & & \text{line 3} \\
f & g & h & i & k & \Xi & l & & & & \text{line 4} \\ \cline{1-7}
 & & M & 7 & N & O & P & Q & & & \text{line 5} \\
 & & m & 7 & n & o & p & q & & & 7 \bullet \partial \\ \cline{3-8}
 & & R & S & T & U & \Sigma & V & W & & \text{line 7} \\
 & & r & s & t & u & 7 & v & w & & \text{line 8} \\ \cline{3-9}
 & & & & X & Y & Z & x & y & z & \text{line 9} \\
 & & & & X & Y & Z & x & y & z & \\ \cline{5-10}
\end{array}
$$

1. The *first* numeral α of the divisor ∂ must be $1,$ since $7\partial,$ as line 6 shows, has six numerals, whereas if $\alpha \geq 2,$ 7∂ would have seven numerals.

Since the remainders in lines 3 and 7 have *six* numerals, $F = 1$ and $R = 1,$ as a result of which $f = r = 1.$ [If $F \geq 2,$ the leading digit in the quotient would be larger than $\kappa,$ because ∂ would "go into" $FGHIK7.$]

Since $\partial \leq 199979$ and $\mu \leq 9,$ the product in line 8 $\mu\partial \leq 1799811,$ and $s < 8.$ $S = 0$ or 9 [from the two lines above], and since there is no remainder in line 9 under s, $S = 0$ and $s = 0$. It also follows from $RS = 10$ that $M = m + 1$ and $m \leq 8$, and from line 6, $7\partial \leq 87nopq$. The problem looks like the following now:

$$
\begin{array}{cccccccccccccccccccccc}
A & B & 7 & C & D & E & L & Q & W & z & \div & 1 & \beta & \gamma & \delta & 7 & \varepsilon & = & \kappa & \lambda & 7 & \mu & \nu \\
a & b & \Delta & c & d & e \\
\hline
1 & G & H & I & K & 7 & L & & & & & & & & & & & & \text{line 3} \\
1 & g & h & i & k & \Xi & l & & & & & & & & & & & & \text{line 4} \\
\hline
& & M & 7 & N & O & P & Q & & & & & & & & & & & \text{line 5} \\
& & m & 7 & n & o & p & q & & & & & & & & & & & 7 \cdot \partial \\
\hline
& & 1 & 0 & T & U & \Sigma & V & W & & & & & & & & & & \text{line 7} \\
& & 1 & 0 & t & u & 7 & v & w & & & & & & & & & & \text{line 8} \\
\hline
& & & & X & Y & Z & x & y & z & & & & & & & & & \text{line 9} \\
& & & & X & Y & Z & x & y & z \\
\hline
\end{array}
$$

2. Since $7 \times 130000 > 87nopq$, $\underline{\beta = 0, 1 \text{ or } 2}$. $\underline{\beta = 0}$ can be <u>eliminated</u> since $9 \times 109979 = 989\,811$ is not a seven digit number, a requirement of line 8.

Now consider $\underline{\beta = 1}$. This requires $\underline{\gamma = 0 \text{ or } 1}$; $\gamma \geq 2$ would result in a "carry" from the product 7γ to $7\beta = 7$, but the second digit (from the left) in 7∂ is 7.

$\underline{\gamma = 0}$, however is <u>impossible</u> as a result of the seven figures in the product in line 8, since even $9 \times 110979 = 998\,811$ has just six digits.

$\underline{\gamma = 1}$ gives $\partial = 111\delta 7\varepsilon$. Line 8 shows that the product $\mu\partial$ is a seven digit number, but $8 \times 111979 = 895\,832$ is a six digit number. Thus $\mu = 9$ in case $\gamma = 1$. If $\varepsilon < 8$, then the hundreds digit in $\mu\partial$ is $9\delta + 6 \bmod 10$ and this is 7 if and only if $\delta = 9$. If $\varepsilon = 8$ or 9, then the hundreds digit in $\mu\partial$ is $9\delta + 7 \bmod 10$ and this is 7 if and only if $\delta = 0$. If $\delta = 0$ however, $\mu\partial \leq 9 \times 111079 = 999\,711$, a six digit number whereas line 8 requires a seven digit number. If $\delta = 9$, line 6 states that $7\partial = 7 \times 11197* = 783***$, contrary to 7 being the second digit (from the left). Thus the case of $\gamma = 1$ is <u>also excluded</u>, and the possibility of <u>$\beta = 1$ must be discarded.</u>

The only suitable value for β is thus $\beta = 2$. Hence $\partial = 12\gamma\delta 7\varepsilon$. $7\partial = m7nopq$ from line 6, and $m \leq 8$, as was established earlier, so it follows that $m = 8$ and $M = 9$. The problem looks like the following now:

$$
\begin{array}{cccccccccccccccccccccc}
A & B & 7 & C & D & E & L & Q & W & z & \div & 1 & 2 & \gamma & \delta & 7 & \varepsilon & = & \kappa & \lambda & 7 & \mu & \nu \\
a & b & \Delta & c & d & e \\
\hline
1 & G & H & I & K & 7 & L & & & & & & & & & & & & \text{line 3} \\
1 & g & h & i & k & \Xi & l & & & & & & & & & & & & \text{line 4} \\
\hline
& & 9 & 7 & N & O & P & Q & & & & & & & & & & & \text{line 5} \\
& & 8 & 7 & n & o & p & q & & & & & & & & & & & 7 \cdot \partial \\
\hline
& & 1 & 0 & T & U & \Sigma & V & W & & & & & & & & & & \text{line 7} \\
& & 1 & 0 & t & u & 7 & v & w & & & & & & & & & & \text{line 8} \\
\hline
& & & & X & Y & Z & x & y & z & & & & & & & & & \text{line 9} \\
& & & & X & Y & Z & x & y & z \\
\hline
\end{array}
$$

3. The *third* figure γ of ∂ can only be 4 or 5, since $7 \times 126070 = 882490$ is greater than line 6, and $7 \times 123979 = 867853$ is smaller than line 6. Moreover, since $9 \times 124070 = 1116630$ is greater than line 8 and $7 \times 125979 = 881853$ is smaller than line 8, $\mu = 8$.

Since $8 \times 124979 = 999832 < 1000000$, the assumption that $\gamma = 4$ fails to satisfy the requirements of line 8, and therefore $\gamma = 5$. The problem looks like the following now:

$$\begin{array}{ccccccccccl}
A & B & 7 & C & D & E & L & Q & W & z & \div\ 1\ 2\ 5\ \delta\ 7\ \varepsilon\ =\ \kappa\ \lambda\ 7\ 8\ \nu \\
a & b & \Delta & c & d & e & & & & & \\ \cline{1-6}
1 & G & H & I & K & 7 & L & & & & \text{line 3} \\
1 & g & h & i & k & \Xi & l & & & & \text{line 4} \\ \cline{1-7}
 & & 9 & 7 & N & O & P & Q & & & \text{line 5} \\
 & & 8 & 7 & n & o & p & q & & & 7 \cdot \partial \\ \cline{3-8}
 & & 1 & 0 & T & U & \Sigma & V & W & & \text{line 7} \\
 & & 1 & 0 & t & u & 7 & v & w & & \text{line 8} \\ \cline{3-9}
 & & & & X & Y & Z & x & y & z & \text{line 9} \\
 & & & & X & Y & Z & x & y & z & \\ \cline{5-10}
\end{array}$$

4. The hundreds digit of $8 \times 125\delta7\varepsilon$ is $8\delta + 5 \bmod 10$ $(\varepsilon \le 4)$ or $8\delta + 6 \bmod 10$ $(5 \le \varepsilon \le 9)$, which is 7 from line 8. Thus $\delta = 4$ or 9. $\delta = 9$ can be eliminated because $7 \times 125970 = 881790$ is greater than line 6, so that only $\delta = 4$ is suitable. It then follows that $\underline{\varepsilon \le 4}$. For each of these ε values $7 \times 12547\varepsilon = 878***$, so $n = 8$. Similarly, $8 \times 12547\varepsilon = 10037**$, so $tu = 03$ from line 8.

Since $\lambda\partial = \lambda \times 12547\varepsilon$ results in a seven digit fourth line, and only 8∂ and 9∂ have seven places, $\underline{\lambda = 8 \text{ or } 9}$. The problem looks like the following now:

$$\begin{array}{ccccccccccl}
A & B & 7 & C & D & E & L & Q & W & z & \div\ 1\ 2\ 5\ 4\ 7\ \varepsilon\ =\ \kappa\ \lambda\ 7\ 8\ \nu \\
a & b & \Delta & c & d & e & & & & & \\ \cline{1-6}
1 & G & H & I & K & 7 & L & & & & \text{line 3} \\
1 & g & h & i & k & \Xi & l & & & & \text{line 4} \\ \cline{1-7}
 & & 9 & 7 & N & O & P & Q & & & \text{line 5} \\
 & & 8 & 7 & 8 & o & p & q & & & 7 \cdot \partial \\ \cline{3-8}
 & & 1 & 0 & T & U & \Sigma & V & W & & \text{line 7} \\
 & & 1 & 0 & 0 & 3 & 7 & v & w & & \text{line 8} \\ \cline{3-9}
 & & & & X & Y & Z & x & y & z & \text{line 9} \\
 & & & & X & Y & Z & x & y & z & \\ \cline{5-10}
\end{array}$$

5. Since $X \ge 1$, it follows that $T \ge 1$, and since $N \le 9$, $T \le 1$; thus $T = 1$. Then $N = 9$ and $X = 1$. Since $2\partial > 200000 > XYZxyz$, it follows that $\nu = 1$ and also that $YZxyz = 2547\varepsilon$. With the results obtained at this point, the problem has the following appearance:

$$A\ B\ 7\ C\ D\ E\ L\ Q\ W\ \varepsilon \div 1\ 2\ 5\ 4\ 7\ \varepsilon = \kappa\ \lambda\ 7\ 8\ 1$$

$$\begin{array}{cccccccccc|l}
a & b & \Delta & c & d & e & & & & & \\
\hline
1 & G & H & I & K & 7 & L & & & & \text{line 3} \\
1 & g & h & i & k & \Xi & l & & & & \text{line 4} \\
\hline
 & & 9 & 7 & 9 & O & P & Q & & & \text{line 5} \\
 & & 8 & 7 & 8 & o & p & q & & & 7 \bullet \partial \\
\hline
 & & 1 & 0 & 1 & U & \Sigma & V & W & & \text{line 7} \\
 & & 1 & 0 & 0 & 3 & 7 & v & w & & \text{line 8} \\
\hline
 & & & & 1 & 2 & 5 & 4 & 7 & \varepsilon & \text{line 9} \\
 & & & & 1 & 2 & 5 & 4 & 7 & \varepsilon & \\
\hline
\end{array}$$

6. From line 8 (8∂), line 6 (7∂), and line 4 ($\lambda\partial$), we have the possibilities:

ε	0	1	2	3	4
vw	60	68	76	84	92
opq	290	297	304	311	318
Ξl when $\lambda = 8$	60	68	76	84	92
Ξl when $\lambda = 9$	30	39	48	57	66

This presents us with ten different (ε, λ) possibilities. If we test each of them by going upward in three successive additions from lines 9 and 8 to line 7, then from lines 7 and 6 to line 5, and finally from lines 5 and 4 to line 3, the only one that gives the required 7 in line 3 is $\varepsilon = 3$ and $\lambda = 8$. In this case $vw = 84$, $U\Sigma VW = 6331$, $opq = 311$, $OPQ = 944$, $ghik\Xi l = 003784$ and $GHIK7L = 101778$. This gives the problem the following appearance:

$$A\ B\ 7\ C\ D\ E\ 8\ 4\ 1\ 3 \div 1\ 2\ 5\ 4\ 7\ 3 = \kappa\ 8\ 7\ 8\ 1$$

$$\begin{array}{cccccccccc|l}
a & b & \Delta & c & d & e & & & & & \\
\hline
1 & 1 & 0 & 1 & 7 & 7 & 8 & & & & \text{line 3} \\
1 & 0 & 0 & 3 & 7 & 8 & 4 & & & & \text{line 4} \\
\hline
 & & 9 & 7 & 9 & 9 & 4 & 4 & & & \text{line 5} \\
 & & 8 & 7 & 8 & 3 & 1 & 1 & & & 7 \bullet \partial \\
\hline
 & & 1 & 0 & 1 & 6 & 3 & 3 & 1 & & \text{line 7} \\
 & & 1 & 0 & 0 & 3 & 7 & 8 & 4 & & \text{line 8} \\
\hline
 & & & & 1 & 2 & 5 & 4 & 7 & 3 & \text{line 9} \\
 & & & & 1 & 2 & 5 & 4 & 7 & 3 & \\
\hline
\end{array}$$

7. Finally, since of all the multiples of ∂, only $5\partial = 627365$ added to 110177 of line 3 gives a number containing a 7 in the third place (from the left), we get $\kappa = 5$. [A spreadsheet is useful here.] At the same time, we conclude that $ab\Delta cde = 627365$ and $ABC7CDE = 737542$, which gives us all of the missing numerals.

7	3	7	5	4	2	8	4	1	3	÷	1	2	5	4	7	3	=	5	8	7	8	1
6	2	7	3	6	5																	
1	1	0	1	7	7	8											line 3					
1	0	0	3	7	8	4											line 4					
		9	7	9	9	4	4										line 5					
		8	7	8	3	1	1										$7 \cdot \partial$					
		1	0	1	6	3	3	1									line 7					
		1	0	0	3	7	8	4									line 8					
				1	2	5	4	7	3								line 9					
				1	2	5	4	7	3													